



Parametric Functions

Studies in Algebra 2
Using
Geometry Expressions





Parametric Functions

Unit Overview

When students see a graph representing projectile motion, it is easy for them to confuse the graph with the actual path of the projectile. That's because the horizontal axis represents the passing of time rather than a horizontal displacement.

Parametric functions provide an easy mechanism for creating a graph that more accurately reflects the physical position of a projectile. Similarly, it portrays circular functions as circles, rather than sine curves.

Still, when looking at a static graph of a parametric function, the sense of movement is lost. Students wonder, "Where is t on the graph?" Computer technology provides the sense of movement, allowing students to connect x , y , and t .

The main goals of this unit are to review functions in general, to familiarize students with parametric functions and their applications, to see how motion is included in the representation of a parametric function, and to uncouple the idea of the vertical line test from the definition of functions in general.

Lesson 1: Functions: A Quick Review

- The definitions of function, control variable, and dependent variable are reviewed. Students use Geometry Expressions to see how the vertical line test relies on the correspondence between x and y on the graph.

Lesson 2: Dude, Where's My Football?

- A problem about the angle required to kick a field goal is used to introduce parametric functions. Geometry Expressions is used to show the motion of a point in two dimensions with respect to time. This naturally evolves into parametric functions. The idea that the parameter, t , only appears on the graph through the motion of the point is emphasized.

Lesson 3: Go Speed Racer

- The idea of motion in the graph of a parametric function is further explored. Functions that have the same x - y graph but different parametric equations are examined. The idea that the vertical line test does not apply to parametric graphs is again stressed.

Lesson 4: Parametric Problems

- Students are asked to apply the parametric function to problems involving projectile motion.



Learning Objectives

This is the first lesson in the unit on parametric functions. Parametric functions are not really very difficult – instead of the value of y depending on the value of x , both are dependent on a third variable, usually t . Confusion can occur when students try to use Cartesian approaches to solve problems with parametric functions. For example, a student might decide that a parametric graph does not represent a function, because it does not pass the vertical line test.

Before we contrast parametric functions with Cartesian functions, we must first review our understanding of Cartesian functions. That is the purpose of this lesson.

Math Objectives

- Review the concept of a function as a mapping from one variable to another.
- Review control variables and dependent variables.
- Review the vertical line test and its role in determining whether a graph is a function.

Technology Objectives

- Use Geometry Expressions to create and graph functions, and constrain points to functions.

Math Prerequisites

- Previous knowledge of the definition and properties of functions is helpful.

Technology Prerequisites

- None

Materials

- Computers with Geometry Expressions.

Overview for the Teacher

1. In part one, students walk through the creation of a parabola in Geometry Expressions, and then placing a point on the parabola.

If students are able to move the point off of the curve, they created a point and then dragged it to the parabola. Thus, it is not part of the parabola. Help them follow the directions more carefully.

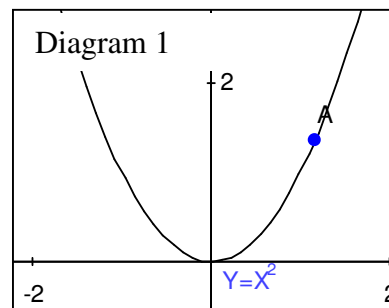
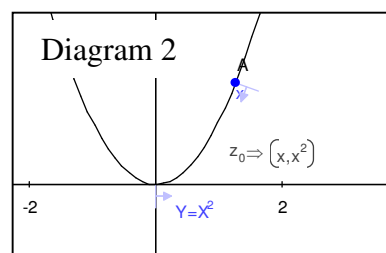


Diagram 1 is representative of student work.

2. When students move the mouse pointer to the left and right, the point will move along the curve. When they move the pointer up and down, the point does not move very much. In fact, the amount it moves reflects the amount the pointer movement varies from the vertical. If student have trouble with this, have them trace the mouse pointer up the y axis.

This is supposed to demonstrate that x is the control variable. This lesson uses the term “control variable” for x , because it fits nicely with the “controls” students use to manipulate the variable in the software. You may wish to connect this term with one of its synonyms: independent variable, input variable, or manipulated variable.



Check to see that students are “controlling” the position of the point by changing the x coordinate.

The results of **Calculate Symbolic Coordinates** will be (x, x^2) .

Diagram 2 shows typical student work.

The second coordinate depends on x .
 y is the dependent variable.

3. Question 3 reinforces the vertical line test. Students drag a vertical line across the parabola. Then, they attach the line to their point, and animate the movement of the point and line together.

You may wish to encourage students to zoom out to see more of the graph before forming a conclusion.

The parabola does pass the vertical line test.



4. In part four, students are presented with several functions they may or may not be familiar with. Students apply the vertical line test to each function.

Note that $\text{sqrt}(x)$ represents $\sqrt{x}$ and that 2^x represents 2^x .

	Coordinates of Point	Pass vertical line test?	Is it a function?
$y = x^2$	(x, x^2)	YES	YES
$y = \sin(x)$	$(x, \sin(x))$	YES	YES
$y = \sqrt{x}$	$(x, \sqrt{x})$	YES	YES
$y = \ln(x)$	$(x, \ln(x))$	YES	YES
$y = 2^x$	$(x, 2^x)$	YES	YES

Note also that the software uses $\log$ for natural log, which is more common in computer and engineering fields.

5. Summary:

Functions in this lesson are sets of points with these properties:

x is the control variable.

y is the dependent variable.

A rule (or formula or equation) explains how x and y are matched up.

Every value for the control variable corresponds with only one value of the dependent variable.

The graph of the function passes the vertical line test.



Name: _____

Date: _____

A Quick Review of Functions

We are about to look at functions in a new way: in a form called “Parametric Functions.” But before we do that, we need to quickly review the old way!

Start by opening a new file with Geometry Expressions. Turn on the axis (look at the icon bar along the top) if it is not already on.

1. Create a parabola.

Click on the **Draw Function** icon



Leave the type as *Cartesian*

Type x^2 in the Y= box

You should see the familiar parabolic graph of $y = x^2$

Now draw a point on the graph.

Click on the **Draw Point** icon



Move your cursor over the parabola. It will change color when you are directly over it. Select the parabola.

Click on the **Draw Select** icon



when you are finished drawing your point.

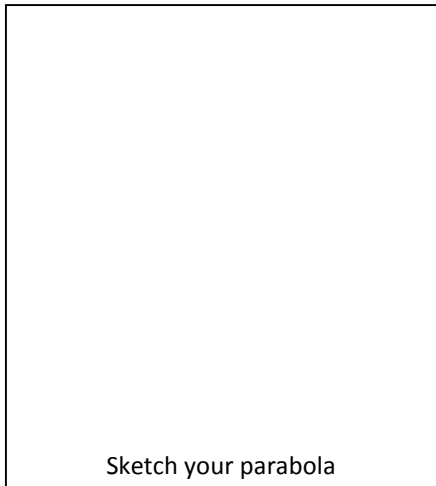
2. Who is in control?

The most important characteristic of a function is that one variable is in control – the control variable – while the other depends on the value of the control variable.

Click on the point and carefully drag your cursor to the left and right. What happens to the point?

Now carefully drag up and down. What happens now?

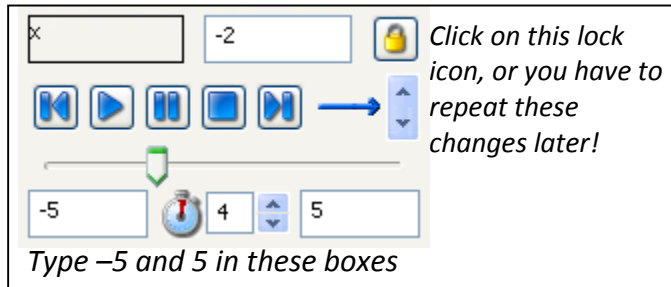
Which variable is in control, x (left and right) or y (up and down)?



Hold down the shift key and select the point, then select the parabola.

Click on **Constrain Point proportional along curve.** 
Change the variable from t to x .

x is now listed in the Variable Tool Panel, though y is not. Select it, and change the animation control box to look like this:



Drag the slider bar, or click on the play button to see how x controls the position of the point.

Now, select the point.

Click on **Calculate Symbolic Coordinates.** 

What are the results?

What does the second coordinate depend on?

Which variable is dependent?

3. It is often said that a graph is a function if it passes the vertical line test. Does the parabola pass the vertical line test?

Click on **Draw Infinite Line**  and draw a line in the window.

Click on the **Select Arrow**  when you are finished drawing the line.

Select the line and Click on **Constrain Direction.** 

Look in the lower right corner of the Geometry Expressions window. It will tell you if you are in degrees or radians.

If you are in degrees, type 90.

If you are in radians, click on the π icon in the symbols window, then type $/2$.

Drag the line across the parabola. Does the line ever cross the parabola more than once?

Hold down shift, select the point, and select the line. Click on **Constrain Incident**.



This will place the point on the vertical line.

Select variable x from the Variables Tool Panel. Click Play.

The definition of a function is this:

A function is a mapping between two sets such that each member of one set corresponds with only one member of the other set.

The vertical line test demonstrates that each value of x corresponds with only one value of y .

Does the parabola pass the vertical line test?

4. Let's look at some other functions.
First make sure you are in Radian mode.
Click Edit on the menu bar.
Click on Preferences.
Click on Math.
Change Angle Mode to Radians.

Look at the Geometry Expressions window and find the equation of your parabola:
 $y = x^2$

Double click on it.
Type $\sin(x)$

What are the coordinates of the point now?

Select x in the variables window and press play.

Does $y = \sin(x)$ pass the vertical line test?

Repeat with $\sqrt{x}$, $\ln(x)$, and 2^x

(Hint: Make sure that your point is in quadrant I or IV (so that $x > 0$) before changing the function equation. Otherwise, it may disappear).

	Coordinates of Point	Pass vertical line test?	Is it a function?
$y = x^2$			
$y = \sin(x)$			
$y = \sqrt{x}$			
$y = \ln(x)$			
$y = 2^x$			

x^2 , $\sin(x)$, $\sqrt{x}$, $\ln(x)$ and 2^x are all **rules** for finding the y coordinate. Every function must include a rule that explains how the control variable is used to calculate the dependent variable.

5. Summary:

Functions in this lesson are sets of points with these properties:

x is the _____ variable

y is the _____ variable

A _____ explains how x and y are matched up.

Every value for the _____ variable corresponds with only one value of the _____ variable.

The graph of the function passes the _____ test.



Learning Objectives

Many students expect a “falling object” graph to look just like the path of the falling object, but that isn’t usually the case. The graph simply shows the height of the object with respect to time. No information about the horizontal displacement is shown on the graph.

One of the virtues of a parametric function is that it gives a “true picture” of the position of the object over time. One of the drawbacks is that the parameter does not appear on the graph.

This drawback is addressed through computer geometry systems, but that’s a topic for the next lesson. We must learn what parametric functions are first.

Math Objectives

- Define a “parametric function.”
- Apply parametric functions in context.

Technology Objectives

- Create parametric functions with Geometry Expressions.
- Constrain points to the graphs of parametric functions to see how they move.

Math Prerequisites

- The concept of *functions*.
- Distance = rate multiplied by time.
- Functions for heights of falling objects.

Technology Prerequisites

- Concept of zooming and panning a display.

Materials

- Computers with Geometry Expressions.



Overview for the Teacher

The introductory problem is not to be solved straight away – its purpose is to create a need for parametric functions. The question “where is the ball” requires a two-part answer: how far has the ball moved horizontally, and how far has it moved vertically.

x stands for the horizontal displacement, and is controlled by t , the amount of time since the ball was kicked.

1. First, the horizontal displacement is addressed. Since nothing influences the horizontal movement of the ball except for air friction (which is disregarded) and impact with the ground (which creates the upper bound of the domain), the function for x is rate * time.
 $X = 20t$.

Following the directions carefully and hitting play will result in point A traveling along the x axis at a uniform velocity.

Responses to “How does t show up on the graph” might be:

Nowhere – only x and y appear on the graph, or
Implicitly in the values for x and y , or
 t appears in the movement of the point.

2. In part 2, the function for vertical displacement is explored.

If students are unfamiliar with falling bodies problems, you will need to step through this part carefully as a class.

The formula for y is $-16t^2 + 30t$

After directions are followed, point B will move up the y axis, slowing to a stop, and then moving down the y axis.

3. Part 3 combines the results of parts 1 and 2 to give a more realistic picture of the path of the football.

Point C will move in the parabolic arc that describes the path of the ball in two dimensions. Students may take one of two approaches to answering the question “where is the ball after 1.5 seconds?” Students may

Move the slider or type in 1.5 for t in the Variable Tool Panel, and then read approximate numbers from the graph or

Substitute 1.5 for t in each formula.

Again, t appears on the graph implicitly in x and y and in the movement of the point.



4. Part 4 introduces the parametric function notation, as implemented by Geometry Expressions.

You will want to show your students the standard format:

$$f(t) = \begin{pmatrix} x = 20t \\ y = -16t^2 + 30t \end{pmatrix}$$

After completing the task, students will see the parabola itself, and the point will move along it.

5. These problems will help reinforce the concepts in this unit.
- 468.8 feet in about 3.85 seconds.
 - About 8.9 feet away from Jackie, and about 12.25 feet high.
 - Yes. The new record will be 76.04 feet.

6. Summary:

A parametric function describes both x and y in terms of a third variable called the parameter.

The parameter is the control variable for a parametric function.

On an x - y graph, the value of t appears only in the motion of the point through the curve.



Name: _____

Date: _____

Dude, Where's my Football?

Jerry is practicing kicking field goals. Jerry kicks the ball so that it has a horizontal velocity of 20 feet per second, and a vertical velocity of 30 feet per second. Where is the ball after 1.5 seconds? (Disregard air friction and other nominal influences!)

In the last lesson, we reviewed functions and found that for a function:

x was in control

y depended on the value of x

What does x stand for in the example above?

Is x in control, or is something controlling x ?

Fill in the blank: The distance the ball has traveled horizontally depends on _____

1. x as a function of t

If you thought that x was controlled by the amount of time since the ball was kicked, then you were absolutely right! x is a function of t .

First, let's find the rule for how x is calculated as a function of time.

Remember that distance = (rate)(time), and that horizontal velocity is 20 feet per second.

Complete the formula: $x =$ _____

Open a new file in Geometry Expressions.

Turn on the axis.

Scale down with the icon on the top icon bar  until the scale includes 50.

Create point A.

Select the point and click on **Constrain Coordinates**.



Type the expression that you wrote in the blank. Use * for multiply.

Type a comma, then a zero for the y coordinate (we'll deal with y in a bit).

Select t from the Variable Tool Panel.

Set the boundaries for t to 0 and 2

Hit play.

What happened to the position of the point?

Does t show up on the graph? How?

2. y as a function of t

The height of the ball also changes with respect to time.
Recall that the function for the height a falling object is

$$y = \frac{1}{2}gt^2 + v_0t + h_0$$

g is the acceleration of gravity, or -32 feet per second squared on earth.

v_0 the initial vertical velocity of the object.

h_0 is the initial height of the object.

Complete the formula for the height of the football:

$y =$ _____

Create point B.

Constrain its coordinates, but this time type 0 for the x coordinate.

Then type a comma,

Then type the expression you've written in the blank.

Remember to use $*$ for multiply, and use 2 for an exponent of two.

Hit play.

What happens to point B?

3. (x, y) as a function of t

Of course, the football is only at one point, not at two. To realistically represent the position of the football, create point C.

Constrain the x coordinate as you did for point A.

Constrain the y coordinate as you did for point B.

Hit play.

Describe what you see.

Where is the ball after 1.5 seconds?

How does t appear on the graph?

4. What path is point C traveling?

The function describing the position of a point (x, y) in terms of a third variable, t , is called a **parametric function**:

$$f(t) = (x, y).$$

t is the control variable.

the point (x, y) is the output of the function.

In other words, one input variable, t , produces two output variables, x and y , as a coordinate pair. Of course, that means that each output variable needs its own rule.

Click on the **Draw function** icon 

Select Parametric for the type.

After $X=$, type $20*t$

After $Y=$, type $-16*t^2+30*t$

Start at 0

End at 2

Click OK

Hit Play.

What do you see? Where is the ball $(x$ and $y)$ after 1.5 seconds?

The ball has traveled _____ horizontally from the point where it was kicked and is _____ feet above the ground.

5. Modify your Geometry Expressions drawing to solve these problems:

- a. A cannonball is fired from a cannon resting on the ground. Its horizontal velocity is 125 feet per second, and its initial vertical velocity is 60 feet per second.

How long does it take for the cannonball to hit the ground?

How far is it from the base of the cannon to the crater the cannonball makes in the ground when it lands?

- b. Jackie shoots a basketball from the free-throw line. She releases the ball from a point 6 feet above the ground. Its horizontal velocity is 14 feet per second, and its initial vertical velocity is 20 feet per second.

How far is the ball from Jackie when it reaches its maximum height?

What is the maximum height of the ball?

- c. In 1990, Randy Byrnes set the world record in the shot put with a throw of 75.85 feet. A competitor releases the shot from a height of 6 feet, with a horizontal velocity of 35 feet per second, and an initial vertical velocity of 32 feet per second.

Will the competitor set a new world record?

6. Summary:

A parametric function describes both _____ and _____ in terms of a third variable called the _____.

_____ is the control variable for a parametric function.

On an x - y graph, the value of t _____.



Learning Objectives

One of the main ideas of the previous lesson is that the control variable t does not appear on the “static” graph of a parametric function. t shows its influence through the speed and direction that the point moves along the path.

Math Objectives

- Use Parametric notation.
- Interpret the effect that T has on the graph as motion.

Technology Objectives

- Use Geometry Expressions to demonstrate motion on a parametric graph.

Math Prerequisites

- Function notation.
- Knowledge of parametric functions, as demonstrated in Lesson 2.
- Some knowledge of circular function would be helpful, but not necessary.

Technology Prerequisites

- No special prerequisites beyond what has been learned in this unit so far.

Materials

- Computer with Geometry Expressions.



Overview for the Teacher

1. Though all of the functions for the first part of the lesson involve $\sin(x)$, only a cursory understanding of $\sin(x)$ is necessary for the lesson – it's just more fun to watch the points move along a sinusoidal curve.
2. Students may run into trouble with constraints if they place the point on the curve, or on the axis. Just have them delete their point and create a new one that is in "white space."
 - a) After constraining the point to $(T, \sin(T))$, the point snaps to the sine curve they have drawn. Animating the point will cause it to move along the curve.
 - b) Point B will also move along the curve, but at twice the speed.
 - c) Changing the parameters of point B to $(T, \sin(2 \cdot T))$ moves the point off the curve – its path has half the period as $\sin(T)$.
 - d) Point B moves along the curve at the same speed as Point A, but from right to left. Both points meet at $(0,0)$.
 - e) $(T^2, \sin(T^2))$: x is ≥ 0 , and the speed changes.

$(\sin(T), \sin(\sin(T)))$ causes the point to oscillate.

There are endless possibilities. The point here is that there is a many-to-one correspondence between parametric functions and their static x-y graphs.

Parameter T appears on the graph through the movement of the point.

3. Part three of the lesson emphasizes the point that the graph does not tell the whole story. It's a demonstration and requires no written response.
4. The vertical line test does not indicate whether a parametric graph is a function because the parameter T does not appear on the graph.
5. The parametric function for part five describes a circle. If students neglected to change the mode to radians earlier in the lesson, they will need to do so at this point. Also, if they neglect to lock the variable T , its start and end value may change unpredictably.

If students are still editing the function when they hit play, the old function is used.

To make the point travel twice as fast, change the function to
$$\begin{pmatrix} X = \cos(2T) \\ Y = \sin(2T) \end{pmatrix}$$

To make the point travel clockwise, students are likely to try
$$\begin{pmatrix} X = \cos(-T) \\ Y = \sin(-T) \end{pmatrix}$$
, based on their results in part 2. If they look closely, they will see that Geometry Expressions has used trigonometric identities to simplify this to:
$$\begin{pmatrix} X = \cos(T) \\ Y = -\sin(T) \end{pmatrix}.$$



To start at the top and move clockwise: $\begin{pmatrix} X = \sin(T) \\ Y = \cos(T) \end{pmatrix}$ Some students will work with phase shifts and negatives signs, which can lead into a nice discussion on trig identities.

To double the radius of the circle, change the function to $\begin{pmatrix} X = 2\cos(T) \\ Y = 2\sin(T) \end{pmatrix}$. Some students will double the circle from the previous question, yielding $\begin{pmatrix} X = 2\sin(T) \\ Y = 2\cos(T) \end{pmatrix}$ or some other variant.

All of the graphs represent functions with respect to T , though their graphs do not pass the vertical line test. The vertical line test is only applicable for functions where x is the control variable.

6. Summary:

A parametric function is a function where the control variable is T and the dependent variables are x and y .

Another name for the control variable is the parameter.

The control variable appears on the graph as motion.

The vertical line test does NOT show whether a parametric curve is a function.

Extension:

One possible solution:

Second hand: $\begin{pmatrix} x = 3\sin(T) \\ y = 3\cos(T) \end{pmatrix}$

Minute hand: $\begin{pmatrix} x = 2\sin\left(\frac{T}{60}\right) \\ y = 2\cos\left(\frac{T}{60}\right) \end{pmatrix}$

Hour hand: $\begin{pmatrix} x = \sin\left(\frac{T}{3600}\right) \\ y = \cos\left(\frac{T}{3600}\right) \end{pmatrix}$

The domain of T will need to be increased to about 377 (120π) to see the entire graph. The minimum value, maximum value, animation duration in the Variable Tool Panel will need to be adjusted. Note that the maximum for animation duration is 60 seconds.



Name: _____

Date: _____

Go Speed Racer

The location of a point on a plane can be expressed with two separate functions, one for the x coordinate and one for the y coordinate. This combination of functions is called a “parametric function.” A third variable, usually T, is the control variable for both functions. T is called “the parameter.”

In this lesson, we’ll see what happens if you make changes to the parameter, while keeping the functions otherwise the same.

1. Setting up

Open a new file in Geometry Expressions. Turn on the axis by clicking the axis button. 

Click on Edit on the menu bar, and select Preferences.

Click Math, and make sure that angle mode is radians.

Draw function. 

Choose **Cartesian** for **Type**.

Type in $\sin(x)$

As you do the lesson, you may wish to zoom in or out to see more of the graph. Use Scale

Up and Scale Down icons on the top icon bar.  .

2. What parametric function will travel along this path?

a. **Draw** Point A, not on the curve.

Constrain its coordinates to $(T, \sin(T))$

What happened to the point?

In the Variables Tool Panel (see diagram 1).

Select T.

Set its minimum value to -8 and its maximum value to 8 .

Set the animation duration to 10.

Click on the lock icon.

Hit Play.

What do you observe?

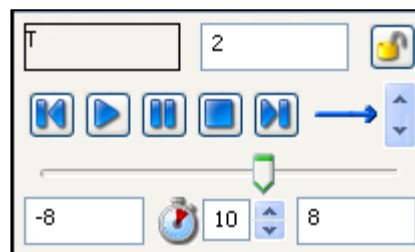


Diagram 1

- b. Draw another point, Point B.
Constrain its coordinates to $(2 \cdot T, \sin(2 \cdot T))$.

Hit Play.

As you observe the two points, what is the same?

What is different?

- c. Change the coordinates of Point B to $(T, \sin(2 \cdot T))$.
Hit Play.

What is the same?

What is different?

- d. Change the coordinates of Point B to $(-T, \sin(-T))$.
What do you think will happen?

Hit Play.

What did happen?

- e. Try changing the coordinates of Point B to $(T^2, \sin(T^2))$.

Try changing the coordinates of Point B to $(\sin(T), \sin(\sin(T)))$.

Try changing the coordinates of Point B to something else that will have the same path as Point A.

How many different parametric functions share the same set of points as $y = \sin(x)$?

How do they differ?

How does the parameter T appear on the graph?

3. Graphs of parametric functions don't tell the whole story.

Create a new file in Geometry Expressions.

Make sure the axis is turned on and that angles are measured in radians.

Click on **Draw Function**.

Change the **Type** to **Parametric**.

Type in $X = 2 \cdot T$

And $Y = \sin(2 \cdot T)$

Click Enter.

The graph shows all the points in the graph *but not their speed or direction*.

To see those features, follow these steps:

Click on **Draw Point**

Click on the graph.

Click on the select icon.



Press SHIFT and select both the point and the curve.

Click on **Constrain Point proportional along curve**.

Type in the Parametric variable T .

Click on play.

The graph of a parametric function does not tell the whole story. You also need to describe the motion of the point on the graph.

4. On two of the last examples in part 2, points are repeated more than once.

Since a parabola is a function, it passes the vertical line test. That's because every value of x corresponds with exactly one value of y .

Values of y can correspond with more than one value of x – that's ok, because x doesn't depend on y ; y depends on x .

You can use the vertical line test on a Cartesian (x - y) graph because it *shows* how each x value corresponds with just one y value.

For parametric functions, x and y both depend on T . They do not depend on each other.

Will the vertical line test (checking how x corresponds with y) work for parametric functions?

Why or why not?

5. Can you control speed and direction?

Open a new file in Geometry Expressions.

Draw a parametric function:

$$X = \cos (T)$$

$$Y = \sin (T)$$

Constrain a point proportional to the curve. Name the constraint t .

Select T from the Variable Tool Panel.

Set its minimum value to 0.

Set its maximum value to 6.28 (that's about 2π).

Lock the variable.

Animate the point.

Describe the path of the point.

You can edit the equations for the function by double-clicking on them.

Try to change the equations so that the point goes around twice as fast.

Record your parametric function here:

Change the equations so the point goes clockwise, and record your parametric function here:

Change the equation so that the point starts at the top of the circle and moves clockwise.
Record your function here:

Change the equations so that the circle's radius is doubled. Record your function here:

Do the graphs represent functions with respect to T ?

Do the graphs pass the vertical line test?

6. Summary:

A parametric function is a function where the control variable is _____ and the dependent variables are _____ and _____.

Another name for the control variable is the _____.

The control variable appears on the graph as _____

The _____ test does NOT show whether a parametric curve is a function.

Extension: Create the Parametric function for a clock.

Create three parametric functions.

The point on one of the functions will travel like the second hand on a clock.

The point on the second function will travel like the minute hand on a clock.

The point on the third function will travel like the hour hand on a clock.

Try to set up the Variable Tool Panel so that the hands move at the correct speeds.



Learning Objectives

The purpose of this lesson is to apply parametric functions to solving problems in context. Students are expected to solve the problems approximately with technology. Solving the problems algebraically would be a beneficial follow-on activity, but is not included in this lesson.

It is recommended that these problems be done collaboratively, and that brainstorming is encouraged rather than offering a particular method.

Hints are included, but may be withheld at teacher discretion.

Math Objectives

- Model problems in context with parametric functions.
- Problem solve collaboratively in groups.

Technology Objectives

- Use Geometry Expressions to model physical behavior.

Math Prerequisites

- Distance = rate * time and related linear models.
- Formulas for falling/dropped projectile motion.
- Parametric equations for a circle, as learned in this unit.
- Familiarity with parametric equations, as learned in this unit.

Technology Prerequisites

- Geometry Expressions skills, as learned in this unit.

Materials

- Computers with Geometry Expressions.

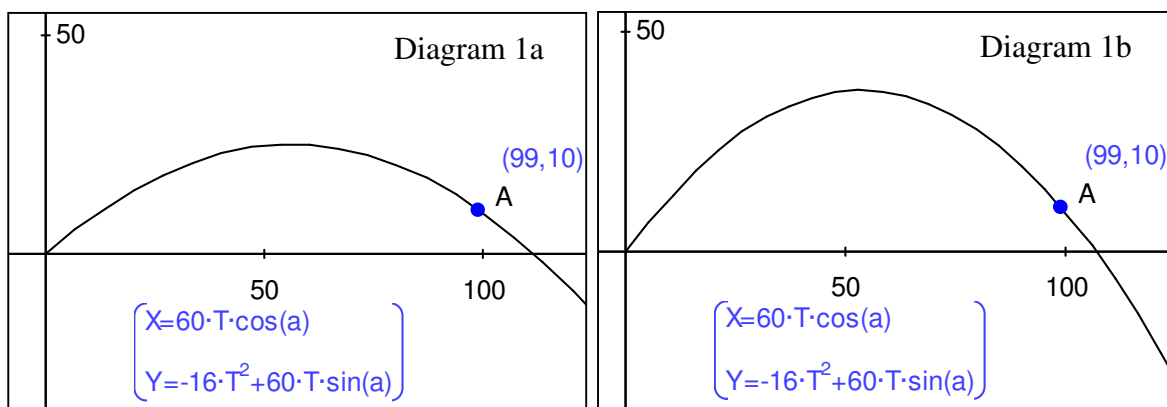
Overview for the Teacher

Students should use this time to problem solve rather than use a prescribed algorithm. There are hints provided at the end of the worksheet. You may wish to withhold the hints or only use a subset, depending on the level of your students.

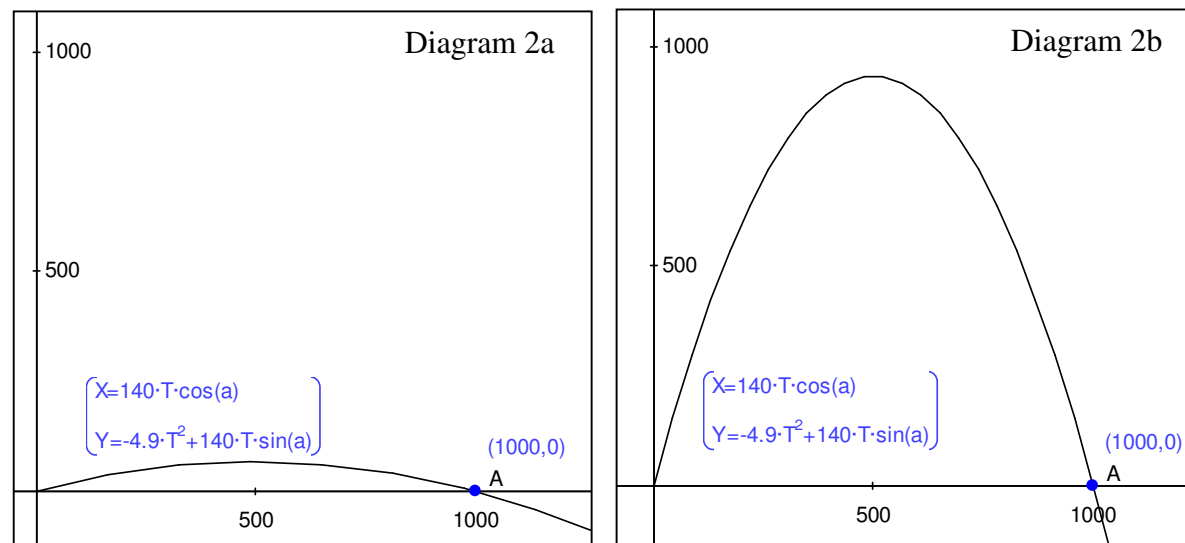
Most of the problems include breaking an initial velocity down into horizontal and vertical components. It is not the intent of this lesson to describe this process, as that may not fall nicely into your sequencing. Instead, “fill-in-the-parameter” type functions are included in the hints portion. You may wish to go into this topic in depth before you begin the lesson, or you may wish to leave this topic in a “mysterious state” until a more appropriate time.

Answers provided here are approximate, since it is not the intent of this lesson to use algebraic techniques to obtain exact answers, although that may be a useful follow-on activity.

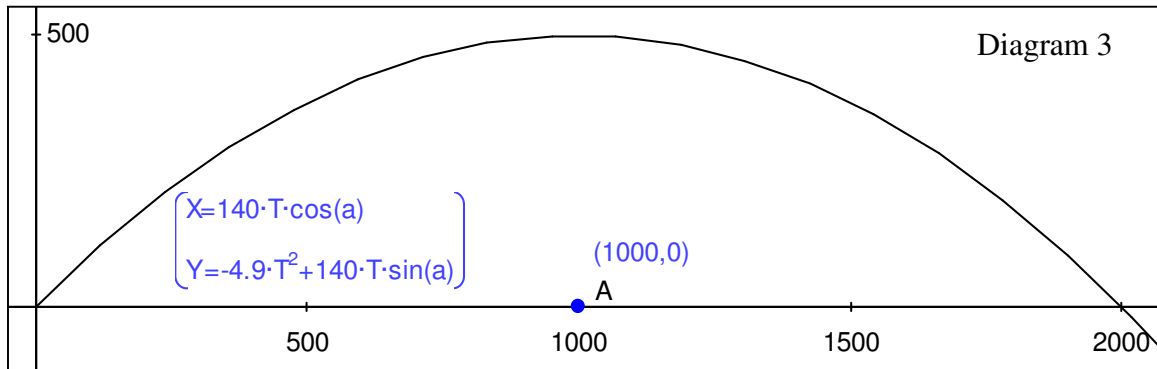
- Angles between 42° and 54° will go over the goalposts. Diagrams 1a and 1b show graphs and formulae.



- The pirate ship should aim at either 15° or 75° . 15° would probably be more effective since it is more of a “direct hit.” Diagrams 2a and 2b show the sketch and parametric function.

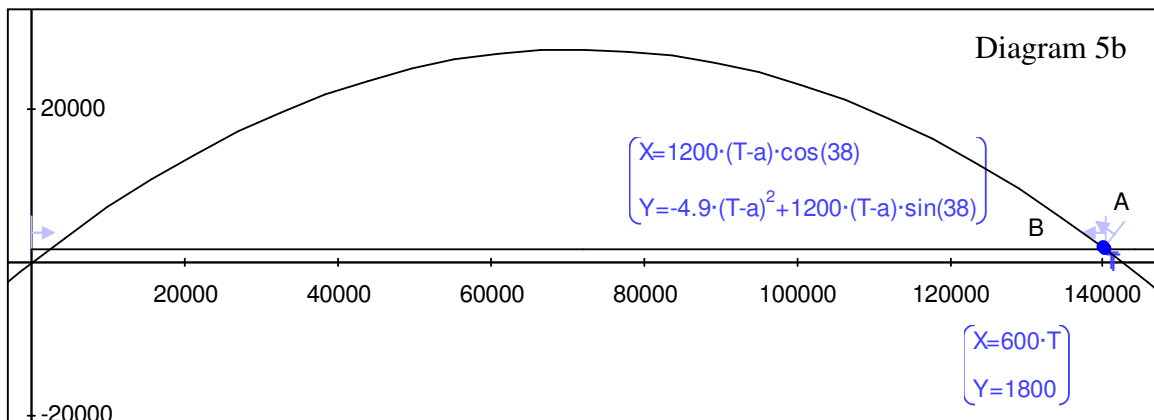
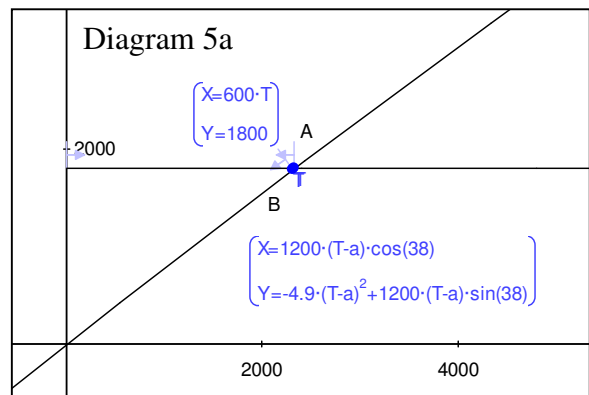
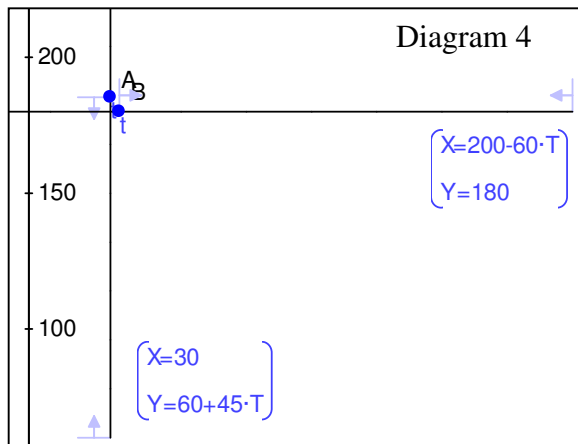


3. The maximum range of the cannon is about 2000 meters (at an angle of 45°). See Diagram 3.



4. The two cars do not collide. See Diagram 4.
5. The gunner may fire at either 1.43 seconds or at 86 seconds. The second answer would allow too much time for course change or other interference. See Diagrams 5a and 5b.

Some versions of Geometry Expressions include a bug that will affect this problem. If students are using Draw Function, their angle measures of 38 degrees will be mistranslated. They can work around the bug by clicking on the equation, and re-typing 38.



6. This problem involves some inductive reasoning. Answers are as follows:

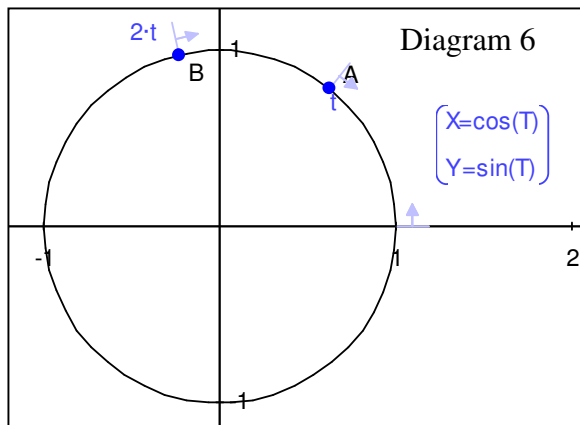
Jogger: 360° or one revolution.

Runner: 180° and 360° .

Biker: 120° , 240° , and 360° .

Mathematician: $\frac{360^\circ}{n-1}, 2\left(\frac{360^\circ}{n-1}\right), 3\left(\frac{360^\circ}{n-1}\right) \dots (n-1)\left(\frac{360^\circ}{n-1}\right)$

See Diagram 6 for a sketch.





Name: _____

Date: _____

Parametric Problems

Use Geometry Expressions and parametric functions to solve these problems.
Record your parametric function and a sketch along with your solution

Hints for some of the problems are at the end of the handout.

1. A football player is attempting a 33-yard field goal. He can kick the football with a maximum initial velocity of 60 feet per second (See the hint section if you need help breaking this down into horizontal and vertical velocities!). If the goal posts are 10 feet above the ground, at about what range of angles must he kick the ball if he is to clear the goal posts?
2. A pirate ship is firing its cannon at a giant squid. The squid is floating motionless, almost tauntingly, 1000 meters away. Cannonballs fly with a velocity of 140 m/sec. Find two cannon angles that will lead to a direct hit. Which of the two angles do you think would be more effective?
3. What is the maximum range of the cannon in problem 2? Why?
4. A car is heading north from a point with coordinates (30, 60) at a speed of 45 miles per hour. A second car is heading west from a point with coordinates (200, 180) at a speed of 60 miles per hour. Do the cars collide?

5. An enemy spy plane is spotted 1800 meters directly over an anti-aircraft gun. The plane is traveling at 600 meters per second. The gunner has time to rotate the gun turret, but no time to change the angle of the barrel, which is set at 38 degrees. The missile will travel with an initial velocity of 1200 meters per second. How long must the gunner wait before firing the missile, if he is going to shoot down the plane? This problem has two solutions. Why would you disregard the second answer?
6. A person is walking one lap around a circular track. Another person is jogging at twice the speed of the walker. How many degrees around the track does the walker turn before the jogger passes him? A runner is traveling at three times the speed of the walker. At what points does the runner pass the walker? A biker is traveling at four times the speed of the walker – when does the biker pass the walker? A theoretical mathematician is traveling at n times the speed of the walker. How many times, and at which angles does the mathematician pass to the walker in one lap for the walker?

Hints

1. The parametric function for a falling or thrown object subject to gravity is:

$$\begin{cases} x = v_0 (\cos \theta) t + x_0 \\ y = \frac{g}{2} t^2 + v_0 (\sin \theta) t + y_0 \end{cases}$$

The object is traveling at an initial velocity of v_0 at an angle of θ with the horizontal.
 g is equal to -9.8 meters per second squared or -32 feet per second squared.
 (x_0, y_0) is the initial position of the object.

Make sure the angle measure is in degrees.

Select Edit from the menu bar and click on Preferences.

Click on the Math icon on the left.

Choose Degrees for angle mode.

After you type in your equation, Geometry Expressions will sometimes insist that you meant radians! If this is the case, click on the equations, and re-type them.

Create a point and constrain it to (99, 10). This will mark the low point of the goalposts.
Change θ so the parabola goes above the point.

2. Read carefully – this problem is in meters. Use $g = -9.8$ meters per second squared. You can modify your Geometry Expressions drawing from problem 1 to solve this problem.
3. By changing θ , what is the largest x intercept you can get? This is the solution.
4. Your point of view on this problem is that you are looking down at the ground (like a map). If a point is moving due North, South, East, or West, then one of its equations will be a constant. South and West will have negative velocities.
5. Your point of view on this problem is from the ground, a distance away from the anti-aircraft gun. The plane is flying from the left to the right.
You can delay the missile by using $(t - k)$ instead of t for the parameter. Alternatively, you can try constraining a point proportional to the curve, and then typing $(t - k)$ for the constraint. Modify k to change the time you wait before firing.
6. Remember the parametric equation for a circle from the last lesson. What did you do to get different speeds?